

# ON THE INTEGRAL CLOSURE OF PLANAR IDEALS

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**ABSTRACT.** We provide an algorithm that computes a set of generators for the integral closure of any ideal  $\mathfrak{a} \subseteq \mathbb{C}\{x, y\}$ . More interestingly, these generators admit a presentation as monomials in a set of maximal contact elements associated to the minimal log-resolution of  $\mathfrak{a}$ .

Let  $(X, O)$  be a germ of smooth complex surface and  $\mathcal{O}_{X,O}$  the ring of germs of holomorphic functions in a neighborhood of  $O$ , which we identify with  $\mathbb{C}\{x, y\}$  by taking local coordinates, and let  $\mathfrak{m}$  the maximal ideal at  $O$ . The aim of this work is to provide an algorithm to compute the integral closure of any ideal  $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ . There are some general algorithms to compute the integral closure as those proposed by Vasconcelos [22, 23], de Jong [10], Leonard and Pelikaan [17] or Singh and Swanson [21] which have been implemented in computer algebra systems such as Macaulay2 [16], Singular [9] or Magma [6]. However, even for rather small examples of planar ideals, the computations are intractable due to the complexity of these algorithms.

In this work we will take a completely different approach using a log-resolution of the ideal and applying some available techniques in the theory of singularities. More precisely, given a log-resolution  $\pi : X' \rightarrow X$  of the ideal  $\mathfrak{a}$ , let  $F$  be an effective Cartier divisor such that  $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ . Then, the integral closure  $\bar{\mathfrak{a}}$  is the stalk at  $O$  of the pushforward  $\pi_* \mathcal{O}_{X'}(-F)$ .

In [2] we provided an effective algorithm that, given a set of generators of the ideal  $\mathfrak{a}$ , computes the divisor  $F$  corresponding to a minimal log-resolution. The main goal of this work is to compute a set of generators for the pushforward  $\pi_* \mathcal{O}_{X'}(-F)$ . Our approach uses two main ingredients: the Zariski decomposition of complete ideals into simple ones, and subtle properties in the theory of adjacent ideals, which relay on results obtained in the study of sandwiched singularities and the Nash conjecture of arcs on these singularities.

Our main result is an algorithm (see Algorithm 2.3) that produces a set of generators of  $\bar{\mathfrak{a}}$ , which we briefly describe. We start fixing a set  $\{f_0, \dots, f_g\}$  of *maximal contact elements* associated to  $\pi$ . These are irreducible curves that are geometrically adapted to the log-resolution of the ideal in the sense that they are parameterized by the dead-ends of the dual graph of the divisor  $F$  (see Subsection 1.4 for details). Once we have this fixed set we consider the decomposition  $F = \rho_1 F_1 + \dots + \rho_r F_r$  of the divisor  $F$  corresponding to the Zariski decomposition  $\bar{\mathfrak{a}} = \mathfrak{a}_1^{\rho_1} \dots \mathfrak{a}_r^{\rho_r}$ , where the ideals  $\mathfrak{a}_i = \pi_* \mathcal{O}_{X'}(-F_i)$  are simple and complete (see [24], [8] for details). For each ideal  $\mathfrak{a}_i$  we compute in a very specific

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unique way an *adjacent ideal below*  $\mathfrak{a}_i$ , i.e. a codimension one ideal  $\mathfrak{b}_i \subsetneq \mathfrak{a}_i$ , and we prove in Proposition 3.4 that we may pick a maximal contact element (or a power of it) belonging to  $\mathfrak{a}_i$  but not belonging to  $\mathfrak{b}_i$ . These adjacent ideals are no longer simple so we may repeat the same procedure. This process finishes after finitely many steps since these adjacent ideals have *smaller support* (see Lemma 3.1). In the end we obtain a tree of ideals with leaves corresponding to the maximal ideal. From the elements of maximal contact that we pick at each step we construct a system of generators of  $\bar{\mathfrak{a}}$ .

As an outcome of our method we obtain a set of generators of the integral closure of a planar ideal given by monomials in this fixed set of maximal contact elements (see Theorem 3.5). That is, we obtain an expression of the form  $\bar{\mathfrak{a}} = (\mathbf{f}^{\alpha_1}, \dots, \mathbf{f}^{\alpha_r})$ , where  $\mathbf{f}^{\alpha} = f_0^{a_0} \cdots f_g^{a_g}$  for  $\alpha = (a_0, \dots, a_g) \in \mathbb{Z}_{\geq 0}^{g+1}$ . More interestingly, if we fix another set of maximal contact elements  $\{f'_0, \dots, f'_g\}$ , the algorithm returns the same monomial expression (in this new set) for the generators of  $\bar{\mathfrak{a}}$ . Actually, the monomial expression that we have for  $\bar{\mathfrak{a}}$  also works for the integral closure of any other ideal  $\mathfrak{b} \subseteq \mathcal{O}_{X,O}$  whose associated divisor  $G$  has the same dual graph as  $F$ .

Finally, we would like to mention that Casas-Alvero [7] proposes a geometrical procedure to obtain a minimal set of generators of the integral closure  $\bar{\mathfrak{a}}$  based on a construction of a filtration by complete ideals, namely a chain of consecutive adjacent ideals between  $\bar{\mathfrak{a}}$  and  $\mathfrak{m}\bar{\mathfrak{a}}$ . The main drawback in order to computationally implement this method is that one does not know a priori which log-resolution of  $\mathfrak{a}$  will dominate all the log-resolutions of each complete ideal appearing in the filtration, since it depends on the choices of the adjacent ideal at each step. In contrast, our procedure always keeps working with complete ideals whose log-resolution is dominated by the log-resolution of  $\mathfrak{a}$ , which is fixed at the beginning.

The algorithm developed in this paper has been implemented in the computer algebra system Magma [6] and it is available at

[github.com/gblanco92/IntegralClosureDim2](https://github.com/gblanco92/IntegralClosureDim2)

## 1. PRELIMINARIES

The aim of this section is to introduce all the background that we will use throughout this work.

**1.1. Log-resolutions and infinitely near points.** Let  $X$  be a smooth complex surface and  $\mathcal{O}_{X,O} \cong \mathbb{C}\{x, y\}$  the ring of germs of holomorphic functions in a neighborhood of a smooth point  $O \in X$  and  $\mathfrak{m} = \mathfrak{m}_{X,O} \subseteq \mathcal{O}_{X,O}$  the maximal ideal at  $O$ . Given a proper ideal  $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$  we have a decomposition  $\mathfrak{a} = (a) \cdot \mathfrak{a}'$ , where  $a \in \mathcal{O}_{X,O}$  is the greatest common divisor of the elements of  $\mathfrak{a}$  and  $\mathfrak{a}'$  is  $\mathfrak{m}$ -primary.

Let  $\pi : X' \rightarrow X$  be a minimal *log-resolution* of  $\mathfrak{a}$ , i.e.  $X'$  is smooth, there exists some effective Cartier divisor  $F$  such that  $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ , and  $F + E$  is a divisor with simple normal crossings support, where  $E = \text{Exc}(\pi)$  is the exceptional locus. We point out that the integral divisor  $F$  has a decomposition  $F = F_{\text{exc}} + F_{\text{aff}}$  into its *exceptional* and *affine* part according to its support. Indeed,  $\mathfrak{a}$  is  $\mathfrak{m}$ -primary whenever  $F$  has only exceptional support, and  $F_{\text{aff}}$  is the strict transform of  $\xi$ , where  $\xi : a = 0$  is the germ of

curve defined by the gcd of the elements of  $\mathfrak{a}$ . For simplicity we will assume in this section that the ideal  $\mathfrak{a}$  is  $\mathfrak{m}$ -primary.

This log-resolution is achieved as a sequence of blow-ups

$$\pi : X' = X_{r+1} \rightarrow X_r \rightarrow \cdots \rightarrow X_1 = X$$

with  $X_{i+1} = \text{Bl}_{p_i} X_i$  for a point  $p_i \in X_i$ . The set  $K$  of points which have been blown up gives a parameterization of the exceptional components  $\{E_p\}_{p \in K}$ . We say that these points are a cluster of *infinitely near to the origin* and we may establish a *proximity relation* between them. Namely, we say that a point  $q \in K$  is *proximate* to  $p \in K$  if and only if  $q$  belongs to the exceptional divisor  $E_p$  corresponding to  $p$  as proper or infinitely near point. We will denote this relation as  $q \rightarrow p$  and we collect all these relations by means of the *proximity matrix*  $P = (P_{p,q})$  defined as:

$$P_{p,q} := \begin{cases} 1 & \text{if } p = q, \\ -1 & \text{if } q \rightarrow p, \\ 0 & \text{otherwise.} \end{cases}$$

The proximity matrix is related to the negative-definite *intersection matrix*  $N = (E_p \cdot E_q)$  by the formula  $N = -P^\top P$ . An infinitely near point  $q$  is proximate to just one or two points. In the former case we say that  $q$  is a *free point*, and in the later it is a *satellite point*. Besides, one can establish a *partial ordering* in  $K$ . Namely,  $q \leq p$  if and only if  $p$  is infinitely near to  $q$ , that is if  $q$  appears before  $p$  in the resolution process.

**1.2. Divisor basis.** Let  $\Lambda_\pi := \bigoplus_{p \in K} \mathbb{Z}E_p$  be the lattice of integral divisors in  $X'$  with exceptional support. We have two different basis of this  $\mathbb{Z}$ -module given by the *strict transforms* and the *total transforms* of the exceptional components. For simplicity we will also denote the strict transforms by  $E_p$  and the total transforms by  $\overline{E}_p$ . In particular, any divisor  $D \in \Lambda_\pi$  can be presented in two different ways

$$D = \sum_{p \in K} v_p(D) E_p = \sum_{p \in K} e_p(D) \overline{E}_p,$$

where the weights  $v_p(D)$  (resp.  $e_p(D)$ ) are the *values* (resp. *multiplicities*) of  $D$ . The relation between values and multiplicities is given by the proximity relations

$$(1.1) \quad v_q(D) = e_q(D) + \sum_{q \rightarrow p} v_p(D).$$

that provide a base change formula  $\mathbf{e}^\top = P \cdot \mathbf{v}^\top$ , where we collect the multiplicities and values in the vectors  $\mathbf{e} = (e_p(D))_{p \in K}$  and  $\mathbf{v} = (v_p(D))_{p \in K}$ , respectively.

Aside from the total and strict transform basis  $\{\overline{E}_p\}_{p \in K}$  and  $\{E_p\}_{p \in K}$  of the lattice  $\Lambda_\pi$  of exceptional divisors, we may also consider the *branch basis*  $\{B_p\}_{p \in K}$  defined as the dual of  $\{-E_p\}_{p \in K}$  with respect to the intersection form. Any divisor  $D \in \Lambda_\pi$  has a presentation

$$(1.2) \quad D = \sum_{p \in K} \rho_p(D) B_p,$$

where  $\rho_p(D) = -D \cdot E_p$  is the *excess* at  $p$  and the relation between excesses and multiplicities is given by  $\boldsymbol{\rho}^\top = P^\top \mathbf{e}^\top$ , where  $\boldsymbol{\rho} = (\rho_p(D))_{p \in K}$  denote the vectors of excesses.

*Remark 1.1.* Let  $\xi : f = 0$  be the germ of curve defined by an element  $f \in \mathcal{O}_{X,O}$  satisfying that  $\pi$  dominates a log-resolution of the principal ideal  $(f)$ . The *total transform* of  $\xi$  is the pull-back  $\bar{\xi} := \pi^*f$  and its *strict transform*  $\xi'$  is the closure of  $\pi^{-1}(\xi - \{O\})$ . Then we have a presentation

$$\bar{\xi} = \xi' + \sum_{p \in K} v_p(f) E_p = \xi' + \sum_{p \in K} e_p(f) \overline{E_p} = \xi' + \sum_{p \in K} \rho_p(f) B_p,$$

where  $v_p(f) := v_p(D)$ ,  $e_p(f) := e_p(D)$  and  $\rho_p(f) := \rho_p(D)$  for  $D = \text{Div}(\pi^*f)_{\text{exc}}$ .

Throughout this work we will be interested on germs of curves associated to branch basis divisors.

**Construction 1.2.** Given a point  $p \in K$ , let  $f_p \in \mathcal{O}_{X,O}$  be an irreducible element such that its strict transform by the log-resolution  $\pi$  intersects transversely  $E_p$  at a smooth point of  $E$ . We have that  $B_p = \text{Div}(\pi^*f_p)_{\text{exc}}$ . Conversely, any  $g \in \mathcal{O}_{X,O}$  with  $B_p = \text{Div}(\pi^*g)_{\text{exc}}$  is irreducible and its strict transform by the log-resolution  $\pi$  intersects transversely  $E_p$  at a smooth point of  $E$ .

Since the  $f_p \in \mathcal{O}_{X,O}$  are irreducible, the points  $q \in K$  such that  $e_q(B_p) \neq 0$  are totally ordered. Furthermore, the log-resolution of any  $f_p \in \mathcal{O}_{X,O}$ ,  $p \in K$  is dominated by  $\pi$ . Moreover, any  $f$  whose log-resolution is dominated by  $\pi$  is characterized as a product  $f = \prod_{p \in T} f_p^{\rho_p}$  of elements  $f_p$  as in Construction 1.2 indexed over a subset  $T \subseteq K$ .

*Remark 1.3.* Let  $\xi : f = 0$ ,  $\eta : g = 0$  be germs of curves such that  $\pi$  dominates their log-resolutions and set  $D = \text{Div}(\pi^*f)_{\text{exc}}$  and  $C = \text{Div}(\pi^*g)_{\text{exc}}$ . Noether's intersection formula [8, Theorem 3.3.1] gives an expression for their intersection multiplicity at  $O$  as  $[\xi.\eta]_O = D \cdot C$ . Hence,  $v_p(f) = [\xi.\eta_p]_O = D \cdot B_p$ , where  $\eta_p : f_p = 0$ , with  $f_p \in \mathcal{O}_{X,O}$  as in Construction 1.2.

**1.3. Complete ideals and antinef divisors.** Given an effective divisor  $D = \sum_{p \in K} v_p E_p \in \Lambda_\pi$ , we may consider its associated ideal sheaf  $\pi_* \mathcal{O}_{X'}(-D)$ . Its stalk at  $O$  is

$$(1.3) \quad H_D = \{f \in \mathcal{O}_{X,O} \mid v_p(f) \geq v_p \text{ for all } E_p \leq D\},$$

This ideal  $H_D$  is *complete*, see [24], and  $\mathfrak{m}$ -primary since  $D$  has only exceptional support. Complete ideals are closed under all standard operations on ideals, except addition: the intersection, product and quotient of complete ideals is complete.

Recall that an effective divisor  $D \in \text{Div}(X')$  is called *antinef* if  $\rho_p = -D \cdot E_p \geq 0$ , for every exceptional prime divisor  $E_p$ ,  $p \in K$ . This notion is equivalent, in the total transform basis, to

$$(1.4) \quad e_p(D) \geq \sum_{q \rightarrow p} e_q(D), \quad \text{for all } p \in K.$$

These are usually called *proximity inequalities*, see [8, §4.2]. By means of the relation given in Equation (1.3), Zariski [24] establishes an isomorphism of semigroups between the set of  $\mathfrak{m}$ -primary complete ideals in  $\mathcal{O}_{X,O}$  whose log-resolution is dominated by  $\pi$  with the multiplicative structure and the set of antinef divisors in  $\Lambda_\pi$ .

Given a non-antinef divisor  $D$ , one can compute an equivalent antinef divisor  $\tilde{D}$ , called the *antinef closure*, under the equivalence relation that both divisors define the same ideal, i.e.  $\pi_*\mathcal{O}_{X'}(-D) = \pi_*\mathcal{O}_{X'}(-\tilde{D})$ , and via the so called *unloading* procedure. This is an inductive procedure which was already described in the work of Enriques [12, IV.II.17] (see [8, §4.6] for more details). The version that we present here is the one considered in [5].

**Algorithm 1.4.** (*Unloading procedure* [5])

**Input:** A divisor  $D = \sum d_p E_p \in \Lambda_\pi$ .

**Output:** The antinef closure  $\tilde{D}$  of  $D$ .

**Repeat:**

- Define  $\Theta := \{E_p \in \Lambda_\pi \mid \rho_p = -D \cdot E_p < 0\}$ .
- Let  $n_p = \left\lceil \frac{\rho_p}{E_p^2} \right\rceil$  for each  $E_p \in \Theta$ . Notice that  $(D + n_p E_p) \cdot E_p \leq 0$ .
- Define a new divisor as  $\tilde{D} = D + \sum_{E_p \in \Theta} n_p E_p$ .

**Until** the resulting divisor  $\tilde{D}$  is antinef.

A fundamental result of Zariski [24] establishes the unique factorization of complete ideals into simple complete ideals, an ideal being simple if it is not the product of ideals different from the unit. A reinterpretation of this result in a more geometrical context is given by Casas-Alvero in [8, §8.4].

**Theorem 1.5.** [24], [8, §8.4] *Let  $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$  be a complete  $\mathfrak{m}$ -primary ideal and let  $F = \sum_{p \in K} \rho_p B_p$  be the expression of the log-resolution divisor of  $\mathfrak{a}$  in the branch basis then,*

$$(1.5) \quad \mathfrak{a} = \prod_{p \in K} H_{B_p}^{\rho_p},$$

with  $H_{B_p}$  being simple complete ideals for any  $p \in K$ .

In the sequel, we will call *simple divisor* any divisor defining a simple complete ideal. As a corollary of Theorem 1.5, simple divisors in  $\Lambda_\pi$  will always be equal to  $B_p$  for some  $p \in K$ .

**1.4. Maximal contact elements.** Let  $\xi : f = 0$  be a germ of an irreducible element such that  $\pi$  dominates the log-resolution of  $(f)$ . Its equisingularity class, or topological equivalence class, is determined by the set of *characteristic exponents*  $\{m_1/n, \dots, m_g/n\}$ , where  $n = e_O(f)$  introduced in [25]. Then,  $f$  has a Puiseux series of the form

$$(1.6) \quad s(x) = \sum_{\substack{j \in (n_0) \\ m_0 \leq j < m_1}} a_j x^{j/n} + \sum_{\substack{j \in (n_1) \\ m_1 \leq j < m_2}} a_j x^{j/n} + \dots + \sum_{\substack{j \in (n_{g-1}) \\ m_{g-1} \leq j < m_g}} a_j x^{j/n} + \sum_{\substack{j \in (n_g) \\ j \geq m_g}} a_j x^{j/n},$$

where  $m_0 = 0$ , and  $n_i = \gcd(n, m_1, \dots, m_i)$  so that, in particular,  $n_0 = n$  and  $n_g = 1$ .

Another way of describing the equisingularity class is using its *semigroup* (see [26]). Consider  $v_\xi : \mathcal{O}_{X,O} \rightarrow \mathbb{Z}$  the valuation induced by the intersection multiplicity of  $\xi$  with another element  $\zeta : h = 0, h \in \mathcal{O}_{X,O}, h \notin (f)$ . It is defined by  $v_\xi(\zeta) = [\xi, \zeta]_O =$

$\text{ord}_t h(t^n, s(t^n))$ , and does not depend on equation defining the germ  $\zeta$  or the parameterization of  $\xi$ . Then, the semigroup of  $\xi$  is

$$\Sigma(\xi) = \{v_\xi(\zeta) \in \mathbb{Z} \mid \zeta : h = 0, h \in \mathcal{O}_{X,O}, h \notin (f)\},$$

and it is finitely generated. Namely,  $\Sigma(\xi) = \langle n, \check{m}_1, \dots, \check{m}_g \rangle$  where

$$(1.7) \quad \check{m}_i = \sum_{j=1}^{i-1} \frac{(n_{j-1} - n_j)m_j}{n_{i-1}} + m_i, \quad \text{for } i = 1, \dots, g.$$

Throughout this work we will be interested on those elements  $f_i \in \mathcal{O}_{X,O}$  such that  $v_\xi(\gamma_i) = [\xi \cdot \gamma_i]_O = \check{m}_i$ , with  $\gamma_i : f_i = 0$  for  $i = 0, \dots, g$ . They will be called *maximal contact elements* of  $\xi$ . Several choices for each  $f_i$  giving raise to different germs can be made; for instance, if  $\xi$  is not tangent to the  $y$ -axis, then  $f_0 = x$  and  $f_1 = y + a_1x + a_2x^2 + \dots$ . In general, these elements can be explicitly computed. Namely, if the equation defining  $\xi$  is in Weierstrass form, the maximal contact elements correspond to its approximate roots (see [18], [19]). Alternatively, if one has a Puiseux series of  $\xi$  as in (1.6), then the equations of the maximal contact elements  $f_i$  have Puiseux series:

$$(1.8) \quad s_i(x) = \sum_{\substack{j \in (n_0) \\ m_0 \leq j < m_1}} a_j x^{j/n} + \dots + \sum_{\substack{j \in (n_{i-1}) \\ m_{i-1} \leq j < m_i}} a_j x^{j/n} + \dots \quad \text{for } i = 1, \dots, g,$$

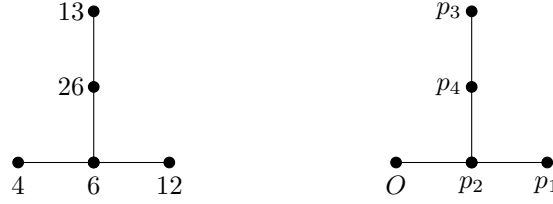
where the non-explicit terms are assumed not to increase the polydromy order, [8, §1.2],  $n/n_{i-1}$  of  $s_i$ , and either  $f_0 = x$  or  $f_0 = y$  depending on whether  $\xi$  is tangent to the  $x$ -axis or the  $y$ -axis respectively, see [8, §5.8]. Notice that the multiplicity at the origin of these maximal contact elements is  $e_O(f_i) = n/n_{i-1}$  for  $i = 1, \dots, g$ .

In general, we will extend the definition of maximal contact elements of an irreducible element of  $\mathcal{O}_{X,O}$  to any ideal of  $\mathcal{O}_{X,O}$  by means of the dual graph of its log-resolution. Unless confusion arises, we will identify each infinitely near point  $p \in K$  and the vertex in the dual graph representing the exceptional component of  $E_p$  associated to it.

The *maximal contact elements* for an ideal  $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$  with divisor  $F$  are those  $f_p \in \mathcal{O}_{X,O}$  with  $p \in K$  considered in Construction 1.2, such that the vertex  $p$  is a dead-end of the dual graph of  $F_{\text{exc}}$ , i.e. the dual graph remains connected when the vertex is removed. In particular, a dead-end vertex will always be a free point or the origin  $O$ , since satellite points are always proximate to two points. A *set of maximal contact elements* contains a unique  $f_p$  for each dead-end vertex  $p \in K$ .

This definition coincides with the one given for an irreducible element  $f \in \mathcal{O}_{X,O}$ . The elements  $f_p \in \mathcal{O}_{X,O}$  such that  $p$  is a dead-end of the dual graph of the log-resolution of  $(f)$  are exactly those with Puiseux series as in (1.8). Conversely, we can construct equations for the maximal contact elements  $f_p$  with  $p \in K$  of any divisor  $F$  parameterized by  $K$ : the coordinates, as points in the projective line,  $q \in E_{q'} \cong \mathbb{P}_{\mathbb{C}}^1$ ,  $q \rightarrow q'$ , for all  $q \leq p$ ,  $e_q(f_p) \neq 0$  determine, and are determined by, the coefficients of a Puiseux series of any  $f_p \in \mathcal{O}_{X,O}$ , see [8, §5.7], which will be as in (1.8).

**Example 1.6.** Let  $\mathfrak{a} = ((y^2 - x^3)^2, x^2y^3) \subseteq \mathcal{O}_{X,O}$  an ideal. The dual graph of the log-resolution of  $\mathfrak{a}$  is



The dead-end points are precisely  $O, p_1, p_3$ . Therefore, a set of maximal contact element for the ideal  $\mathfrak{a}$  is  $\{f_0, f_1, f_2\}$  with

$$\begin{aligned} f_0 &= x + a_{2,0}x^2 + a_{0,2}y^2 + a_{1,1}xy + \cdots, \\ f_1 &= y + b_2x^2 + b_3x^3 + \cdots, \\ f_2 &= y^2 - x^3 + \sum_{3i+2j>6} c_{i,j}x^i y^j, \end{aligned}$$

and different choices of  $a_{i,j}, b_i, c_{i,j} \in \mathbb{C}$  will give different sets of maximal contact elements.

## 2. AN ALGORITHM TO COMPUTE THE INTEGRAL CLOSURE

In this section we present an algorithm which computes a set of generators for the integral closure of a given ideal  $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ . If we consider a decomposition  $\mathfrak{a} = (a) \cdot \mathfrak{a}'$ , where  $a \in \mathcal{O}_{X,O}$  is the greatest common divisor of the elements of  $\mathfrak{a}$  and  $\mathfrak{a}'$  is  $\mathfrak{m}$ -primary, then  $\overline{\mathfrak{a}} = \overline{(a)} \cdot \overline{\mathfrak{a}'} = (a) \cdot \overline{\mathfrak{a}'}$ , since principal ideals are complete. Therefore, without loss of generality, we may assume that  $\mathfrak{a}$  is  $\mathfrak{m}$ -primary. The main property that we are going to use is that the ideal associated to the log-resolution divisor  $F$  of  $\mathfrak{a}$  determines its integral closure, that is  $\overline{\mathfrak{a}} = H_F$ , see [8, §8.3].

We briefly describe the main ideas behind Algorithm 2.3. We start with the effective divisor  $F$  which is always antinef. It decomposes into simple divisors  $F = \rho_{q_1}B_{q_1} + \cdots + \rho_{q_r}B_{q_r}$  with all  $\rho_{q_i} > 0$ . For each simple divisor  $B_{q_i}$  we compute the antinef closure of  $B_{q_i} + E_O$  which we denote  $\widehat{D}_i$ . This divisor describes an *adjacent ideal*  $H_{\widehat{D}_i}$  below  $H_{B_{q_i}}$ , i.e. an ideal  $H_{\widehat{D}_i} \subsetneq H_{B_{q_i}}$  such that  $\dim H_{B_{q_i}}/H_{\widehat{D}_i} = 1$  as  $\mathbb{C}$ -vector space. Indeed, any ideal has many adjacent ideals, but we focus on this particular one. Next, we find, among the set of maximal contact elements of  $\mathfrak{a}$ , an element  $f \in \mathcal{O}_{X,O}$  belonging to  $H_{B_{q_i}}$  but not to  $H_{\widehat{D}_i}$ . Now,  $\widehat{D}_i$  is no longer simple but has smaller support than  $B_{q_i}$  in the total transform basis  $\{\overline{E_p}\}_{p \in K}$ . Therefore we may repeat the same procedure with  $F := B_{q_j}, j < i$  until  $F = B_O := \text{Div}(\pi^*\mathfrak{m})$ .

This first part generates a tree where each vertex is an antinef divisor and where the leafs of the tree are all  $B_O = \text{Div}(\pi^*\mathfrak{m})$ . The second part traverses the tree bottom-up computing in each node the ideal associated to the divisor. Using the notations from the above paragraph, given any node in the tree with divisor  $D$ , the ideal  $H_D$  is computed multiplying the ideals in child nodes and adding the element  $f$  to the resulting generators.

Before giving a more explicit description of the algorithm, let us first state two technical results. The first one presents some properties of adjacent ideals based on results obtained by Fernández-Sánchez (see [13], [14], [15]) in the study of sandwiched singularities and the Nash conjecture of arcs on these singularities.



**Proposition 2.1.** *Let  $H_D$  be the complete ideal defined by an antinef divisor  $D \in \Lambda_\pi$ . Consider  $\widehat{D}$  the antinef closure of  $D + E_O$ , obtained from  $D + E_O$  by unloading on a given<sup>1</sup> subset of points  $T \subseteq K$ . Then,  $H_{\widehat{D}} \subsetneq H_D$  are adjacent ideals if and only if  $\rho_O(D) = 0$ .*

Furthermore, if  $H_{\widehat{D}} \subsetneq H_D$  are adjacent then,

- i)  $\sum_{p \in T} E_p$  is the connected component of  $\sum_{p \in K, \rho_p(D)=0} E_p$  containing  $E_O$ ;
- ii)  $e_O(\widehat{D}) = e_O(D) + 1$ , and  $e_p(D) - 1 \leq e_p(\widehat{D}) \leq e_p(D)$  for any  $p \in K$ ,  $p \neq O$ ;  
moreover  $\rho_O(\widehat{D}) > 0$ ;
- iii) if  $p \in K \setminus T$  and  $p$  is proximate to some point in  $T$  then,  $e_p(\widehat{D}) = e_p(D) - 1$ .

*Proof.* Consider the cluster  $K'$  obtained from  $K$  by adding  $r := \rho_O(D) + 1$  free points  $p_1, \dots, p_r$  lying on  $E_O$ . Let  $\pi' : Y' \rightarrow X$  be the composition of  $\pi$  with the sequence of blow-ups of the points  $p_1, \dots, p_r$ . Denote by  $\overline{G} \in \text{Div}(Y')$  the pullback of any  $G \in \text{Div}(X')$ . For simplicity, denote the strict and the total transform basis by  $\{E_p\}_{p \in K'}$  and  $\{\overline{E}_p\}_{p \in K'}$  respectively in the lattice  $\Lambda_{\pi'}$ .

Clearly, both  $\overline{D} + E_O$  and  $\overline{D} + E_{p_1} + \dots + E_{p_r}$  are not consistent, whereas  $\overline{D} + E_{p_1} + \dots + E_{p_i}$  are consistent for all  $1 \leq i < r$ . Moreover, when applying the unloading procedure described in Algorithm 1.4, we find that the antinef closures of  $\overline{D} + E_O$  and  $\overline{D} + E_{p_1} + \dots + E_{p_r}$  are the same, say it  $\widehat{D}'$ , and  $e_{p_i}(\widehat{D}') = 0$  for all  $1 \leq i \leq r$ . Indeed, the first step of the unloading procedure applied to  $\overline{D} + E_O$  or  $\overline{D} + E_{p_1} + \dots + E_{p_r}$  gives the same divisor  $\overline{D} + E_O + E_{p_1} + \dots + E_{p_r}$ . Furthermore,  $\widehat{D}'$  is the pullback of the antinef closure  $\widehat{D}$  of  $D + E_O$  in  $\text{Div}(X')$  and hence they define the same complete ideal  $H_{\widehat{D}'} = H_{\widehat{D}}$ .

Now, from [8, §4.7], the codimension of a complete ideal  $H_G$  defined by a divisor  $G \in \Lambda_\pi$  satisfies

$$\dim \mathcal{O}_{X,O}/H_G = \sum_{p \in K} \frac{e_p(\tilde{G})(e_p(\tilde{G}) + 1)}{2} \leq \sum_{p \in K} \frac{e_p(G)(e_p(G) + 1)}{2},$$

where  $\tilde{G}$  is the antinef closure of  $G$ . Hence,

$$H_{\widehat{D}} = H_{\overline{D} + E_{p_1} + \dots + E_{p_r}} \subsetneq H_{\overline{D} + E_{p_1} + \dots + E_{p_{r-1}}} \subsetneq \dots \subsetneq H_{\overline{D} + E_{p_1}} \subsetneq H_D$$

is a chain of adjacent complete ideals, giving  $\dim H_{B_{q_i}}/H_{\widehat{D}_i} = r = \rho_O(D) + 1$ . Therefore,  $H_{\widehat{D}} \subsetneq H_D$  are adjacent, if and only if  $\rho_O(D) = 0$ .

Finally, from [3, 2.1] and [13, 4.6] claim i) follows. Claim ii) and iii) are consequences of [14, 4.2] and [15, 2.2] (see also [4, 3.1]). □

**Lemma 2.2.** *Let  $H_D$  be the complete ideal defined by an antinef divisor  $D \in \Lambda_\pi$ . The divisor  $D + \overline{E}_O$  is antinef and  $H_{D + \overline{E}_O} = \mathfrak{m}H_D$ .*

*Proof.* Clearly  $\mathfrak{m} = \{f \in \mathcal{O}_{X,O} \mid e_O(f) \geq 1\}$ . Thus,  $\mathfrak{m} = H_{\overline{E}_O}$ , and since  $\overline{E}_O = B_O$  it is antinef and the result follows by the correspondence between antinef divisor and complete ideals. □

<sup>1</sup> $T$  is the set of points  $p \in K$  that parameterize the support of  $\widehat{D} - (D + E_O)$ .



**Algorithm 2.3.** (*Integral closure of an ideal*)

**Input:** An ideal  $\mathfrak{a} = (a_1, \dots, a_r) \subseteq \mathcal{O}_{X,O}$ .

**Output:** Generators of the complete ideal  $\bar{\mathfrak{a}}$ .

1. Let  $\mathfrak{a} = (a) \cdot \mathfrak{a}'$ ,  $a \in \mathcal{O}_{X,O}$ ,  $\mathfrak{a}'$   $\mathfrak{m}$ -primary and compute the log-resolution divisor  $F \in \Lambda_\pi$  of  $\mathfrak{a}'$  using the algorithm in [2].
2. Compute a set of maximal contact elements  $\{f_i\}_{i \in I}$  of the ideal  $\mathfrak{a}$ . Set  $D^{(0)} := F$  and proceed from step (0.1).

*Step (i):*

- i.1 Decompose  $D^{(i)}$  into  $d_i := \#\{p \in K \mid \rho_p(D^{(i)}) > 0\}$  simple divisors.
- i.2 For each  $j = 1, \dots, d_i$ , consider  $q_j \in \{p \in K \mid \rho_p(D^{(i)}) > 0\}$  and assume  $B_{q_j} = \sum_{p \in K} e_p \bar{E}_p$ .

i.j.1 **Stop at the maximal ideal:**

If  $B_{q_j} = B_O := \text{Div}(\pi^* \mathfrak{m})$ , then set  $H_{B_{q_j}} = (f_{i_0}, f_{i_1})$  for  $i_0, i_1 \in I$  such that they are smooth and transverse at  $O$ , then stop. Otherwise, proceed from i.j.2.

i.j.2 **Compute the adjacent ideal to  $H_{B_{q_j}}$ :**

Perform unloading on the divisor  $B_{q_j} + E_O$  to get its antinef closure  $\hat{D}_j$ .

i.j.3 **Select a maximal contact element in  $H_{B_{q_j}} \setminus H_{\hat{D}_j}$ :**

Let  $p \in K$  the last free point such that  $e_p \neq 0$ . Take  $\tau_j \in I$  such that  $e_p(f_{\tau_j}) = 1$  and  $e_O(f_{\tau_j}) \leq e_O$ . Define the integer  $n_j := e_O/e_O(f_{\tau_j})$ .

i.j.4 **Recursive step:**

Assume  $H_{\hat{D}_j}$  has been computed after performing step (i+1) with  $D^{(i+1)} := \hat{D}_j$ .

i.j.5 **Set:**

$$H_{B_{q_j}} = \left( f_{\tau_j}^{n_j} \right) + H_{\hat{D}_j}.$$

i.3 **Apply Zariski's factorization theorem:**

Compute the product  $H_{D^{(i)}} = \prod_{j=1}^{d_i} H_{B_{q_j}}^{\rho_{q_j}}$ , giving generators  $h_1, \dots, h_{s_i}$ .

i.4 **Set, using Nakayama's lemma:**

$$H_{D^{(i)}} = \left( h_k \mid \pi^* h_k \notin \mathcal{O}_{X'}(-D^{(i)} - \bar{E}_O), k = 1, \dots, s_i \right) \mathcal{O}_{X,O}.$$

3. **Return:**  $\bar{\mathfrak{a}} = (a) \cdot H_F$ .

*Remark 2.4.* In order to clarify some steps of the algorithm we point out the following:

- At steps (i.j.1) and (i.j.3) of Algorithm 2.3 we have to choose maximal contact elements. These choices are not necessary unique as several maximal contact elements may fulfill the required conditions.

- Since the sheaf ideals  $\mathcal{O}_{X'}(-D)$ , with  $D \in \text{Div}(X')$ , are defined by valuations, testing whether the pullback of an element  $f$  belongs to  $\mathcal{O}_{X'}(-D)$  or not is only a matter of comparing the values  $v_p(\text{Div}(\pi^*f))$  and  $v_p(D)$  for all  $p \in K$ .
- It is clear from Nakayama's lemma and Lemma 2.2 that a set of elements of  $\mathcal{O}_{X,O}$  is a system of generators of  $H_D$  if and only if its classes modulo  $H_{D+\overline{E}_O}$  are a system of generators of  $H_D/H_{D+\overline{E}_O}$  as  $\mathbb{C}$ -vector space. Equivalently, any element of  $H_{D+\overline{E}_O}$  is redundant in a system of generators of  $H_D$ .

**Example 2.5.** Consider the ideal  $\mathfrak{a} = ((y^2 - x^3)^2, x^2y^3) \subseteq \mathcal{O}_{X,O}$ . Using the algorithm in [2], we compute the effective antinef divisor  $F = B_{p_4}$  associated to  $\mathfrak{a}$  through its log-resolution, represented using the dual graph appearing in Figure 1. A set of maximal contact elements of  $\mathfrak{a}$  was computed in Example 1.6, for instance  $f_0 = x, f_1 = y, f_2 = y^2 - x^3$ . The steps of Algorithm 2.3 will be illustrated by means of the tree-shaped graph in Figure 1.

The vertices of the tree contain antinef divisors represented using dual graphs. The root node contains the log-resolution divisor of the ideal  $\mathfrak{a}$ . Dashed arrows connect simple divisors  $B_{q_j}$  with its corresponding adjacent  $\widehat{D}_j$  from step (i.j.2). The maximal contact elements from step (i.j.3) that belong to  $H_{B_{q_j}}$  but not to  $H_{\widehat{D}_j}$  are represented next to dashed arrows. Solid arrows connect  $\widehat{D}_j =: D^{(i+1)}$  with each of its irreducible components  $B_p$ , with  $p \in K$ . Finally, the weight  $\rho_p^{(i)}$  of each divisor  $B_p, p \in K$ , in  $\widehat{D}^{(i)}$  is written next to the solid arrows.

We have added some extra indices to the divisors appearing in the algorithm to highlight at which step we encounter them. Hopefully it does not create any confusion since its meaning should be clear from the context. The generators of the ideals associated to the divisors in each intermediate step are then:

- $H_{B_O} = \mathfrak{m} = (x, y)$ .
- $H_{B_{p_1}} = (y) + H_{D_1^{(2)}} = (y) + \mathfrak{m}^2 = (y, x^2, xy, y^2)$ .
- $H_{B_{p_2}} = (y^2) + H_{D_2^{(2)}} = (y^2) + \mathfrak{m}^3 = (y^2, x^3, x^2y, xy^2, y^3)$ .
- $H_{D^{(1)}} = B_O^2 \cdot B_{p_1} \cdot B_{p_2} = (x, y)^2 \cdot (y, x^2, xy, y^2) \cdot (y^2, x^3, x^2y, xy^2, y^3)$   
 $= (x^7, \cancel{x^6y}, \cancel{x^5y^2}, \cancel{x^4y^3}, \dots, x^5y, x^4y^2, \dots, \cancel{x^3y^3}, \cancel{x^2y^4}, x^2y^3, xy^4, y^5)$   
 $= (x^7, x^5y, x^4y^2, x^2y^3, xy^4, y^5)$ .
- $B_{p_4} = ((y^2 - x^3)^2) + H_{D^{(1)}} = ((y^2 - x^3)^2, x^7, x^5y, x^4y^2, x^2y^3, xy^4, y^5)$ .
- $\overline{\mathfrak{a}} = H_F := H_{D^{(0)}} = B_{p_4} = ((y^2 - x^3)^2, x^7, x^5y, x^4y^2, x^2y^3, xy^4, \cancel{y^5})$ .

The crossed out elements are those that are redundant by step (i.4) and Remark 2.4. Observe that, although many crossed out elements are actually multiple of other elements, step (i.4) and Remark 2.4 allows us to remove  $y^5$  which is not multiple of any other element.

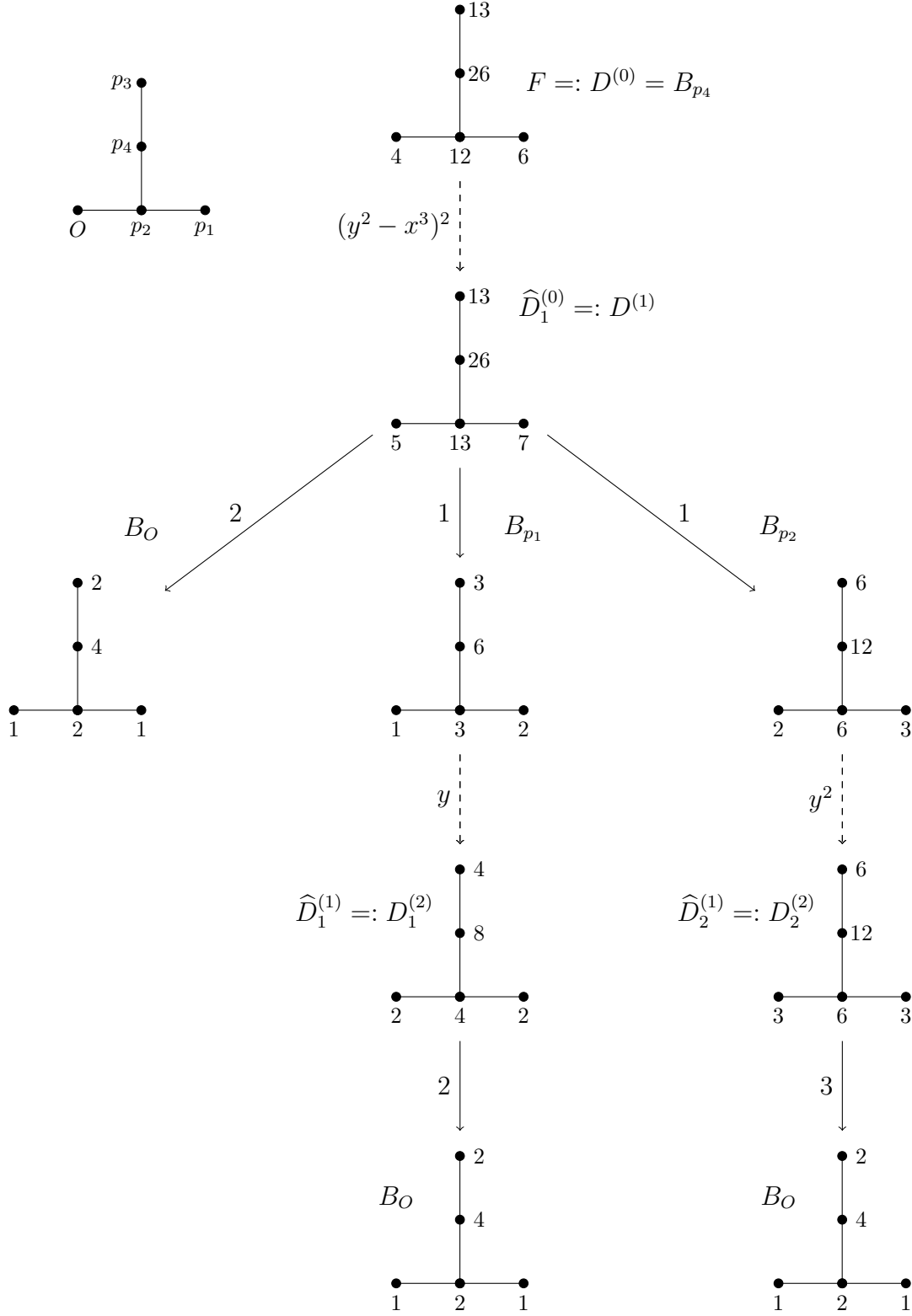


FIGURE 1. Tree of divisors from Algorithm 2.3 in Example 2.5.

*Remark 2.6.* As an outcome of the algorithm, we see that the integral closure of  $\mathfrak{a}$  admits a monomial expression

$$\overline{\mathfrak{a}} = (f_2^2, f_0^7, f_0^5 f_1, f_0^4 f_1^2, f_0^2 f_1^3, f_0 f_1^4),$$

in the set of maximal contact elements  $f_0 = x, f_1 = y, f_2 = y^2 - x^3$  associated to  $\pi$  that we fixed in the beginning. We would get the same monomial expression for any other set of maximal contact elements.

However, we might get a different monomial expression depending on the maximal contact elements (or powers of) that we choose in step (i.j.3) of Algorithm 2.3. In this example, when choosing an element in  $H_{B_{p_3}}$  that does not belong to  $H_{\widehat{D}_2^{(1)}}$  we took  $f_1^2 = y^2$ , but we could also had chosen  $f_2 = y^2 - x^3$ . In this later case the final system of generators is

$$\overline{\mathfrak{a}} = ((y^2 - x^3)^2, x^2 y (y^2 - x^3), x y^2 (y^2 - x^3), x^7, x^5 y, x^4 (y^2 - x^3), x^4 y^2),$$

so we get the monomial expression

$$\overline{\mathfrak{a}} = (f_2^2, f_0^2 f_1 f_2, f_0 f_1^2 f_2, f_0^7, f_0^5 f_1, f_0^4 f_2, f_0^4 f_1^2).$$

### 3. CORRECTNESS OF THE ALGORITHM

In this section we will prove that Algorithm 2.3 developed in Section 2 is correct. First, we need to check that it ends after a finite number of steps. The key point is to prove that the divisor  $\widehat{D}_j$  defining the adjacent ideal to the simple ideal  $H_{B_{q_j}}$  has smaller support in the total transform basis than  $B_{q_j}$ . To that end, let us first introduce some notation. For any  $D \in \Lambda_\pi$  we define

$$|D|_{\overline{E}} := \#\{\overline{E}_p \mid e_p(\widetilde{D}) \neq 0, p \in K\},$$

where  $\widetilde{D}$  is the antinef closure of  $D$ .

**Lemma 3.1.** *Using the notations in Algorithm 2.3, assume that  $B_{q_j}$  is a simple divisor different from  $B_O = \text{Div}(\pi^* \mathfrak{m})$ . Let  $\widehat{D}_j$  be the antinef closure of  $B_{q_j} + E_O$  computed in step (i.j.2). Then,  $\widehat{D}_j$  has smaller support than  $B_{q_j}$  in the total transform basis,  $\{\overline{E}_p\}_{p \in K}$ . That is,*

$$|B_{q_j}|_{\overline{E}} > |\widehat{D}_j|_{\overline{E}}.$$

*Proof.* Since  $B_{q_j} \neq B_O$ , the excess of  $B_{q_j}$  at  $O$  is  $\rho_O(B_{q_j}) = 0$ . Then, according to Proposition 2.1, the ideal defined by  $B_{q_j} + E_O$  is an adjacent ideal below  $H_{B_{q_j}}$ . Let  $T \subseteq K$  be the points on which unloading is performed to obtain the antinef closure  $\widehat{D}_j$  from  $B_{q_j} + E_O$ . By Proposition 2.1,  $T$  are the points  $p \in K$  whose associated exceptional divisor  $E_p$  belongs to the same connected component as  $E_O$  in  $\sum_{O \leq p < q_j} E_p$ . Observe that  $\sum_{O \leq p < q_j} E_p$  has either one or two components, according if  $q_j$  is either free or satellite. In both cases,  $q_j$  is proximate to the point  $p \in T$  whose exceptional divisor cuts  $E_{q_j}$ , i.e.  $E_p \cdot E_{q_j} = 1$ . Hence, invoking Proposition 2.1 again, the multiplicity at  $q_j$  of  $\widehat{D}_j$ , after performing unloading on  $B_{q_j} + E_O$ , decreases by one. Since  $B_{q_j}$  is simple, the multiplicity of  $B_{q_j}$  at  $q_j$  is one. Hence, the multiplicity of  $\widehat{D}_j$  at  $q_j$  is zero, giving the desired result.  $\square$

In the next proposition we prove that Algorithm 2.3 ends after a finite number of steps and the leaves of the tree that we construct correspond to the maximal ideal. To emphasize the dependence of the divisors on a specific step  $(i)$  of the algorithm we will use the notation  $B_{q_j}^{(i)}$  and  $\widehat{D}_j^{(i)}$ .

**Proposition 3.2.** *Algorithm 2.3 ends after a finite number of steps.*

*Proof.* Since, using Equation (1.4), the multiplicities of  $B_p$  decrease, we have  $|B_p|_{\overline{E}} = 1$  for some  $p \in K$  if and only if  $p = O$  and then  $B_p = \text{Div}(\pi^*\mathfrak{m})$ .

Using the notations in Algorithm 2.3, assume that we are in step  $(i)$  and we have a simple divisor  $B_{q_j}^{(i)}$  in step  $(i.j.1)$ . If  $q_j = O$ , then  $B_O = \text{Div}(\pi^*\mathfrak{m})$  and we are done. Otherwise, since  $q_j \neq O$ , we have that  $|B_{q_j}^{(i)}|_{\overline{E}} > |\widehat{D}_j^{(i)}|_{\overline{E}}$  by Lemma 3.1. Since  $D^{(i+1)} := \widehat{D}_j^{(i)}$  admits a decomposition  $D^{(i+1)} = \sum_{p \in K} \rho_p^{(i+1)} B_p^{(i+1)}$ , we have  $|D^{(i+1)}|_{\overline{E}} \geq |B_p^{(i+1)}|_{\overline{E}}$ . Hence,  $|B_{q_j}^{(i)}|_{\overline{E}} > |B_p^{(i+1)}|_{\overline{E}}$ ,  $p \in K$ , and by induction we obtain the desired result.  $\square$

**Lemma 3.3.** *Let  $B_q$  be a branch basis divisor associated to a satellite point  $q \in K$ . Let  $\Sigma(\xi) = \langle n, \check{m}_1, \dots, \check{m}_g \rangle$  be the semigroup of  $\xi : f_q = 0$  and take  $\gamma : f_g = 0$  such that  $[\gamma \cdot \xi]_O = \check{m}_g$ . Then,  $B_q^2 = [\zeta \cdot \xi]_O$  with  $\zeta : f_g^{n_{g-1}} = 0$ . Furthermore,  $v_p(f_g^{n_{g-1}}) \geq v_p(B_q)$ , for all  $p \leq q$ .*

*Proof.* Assume that  $B_q = \sum_{p \in K} e_p \overline{E}_p$ . The first claim follows from the following computation:

$$B_q^2 = \sum_{p \in K} e_p^2 = \sum_{i=1}^g (m_i - m_{i-1}) n_{i-1} = \sum_{i=1}^{g-1} (n_{i-1} - n_i) m_i + n_{g-1} m_g = n_{g-1} \check{m}_g = n_{g-1} [\gamma \cdot \xi]_O = [\zeta \cdot \xi]_O.$$

where the second equality is true since  $q \in K$  is a satellite point (see [8, §5.10]) and the fourth equality comes from (1.7).

To prove the second claim, in virtue of Remark 1.3, it suffices to check the inequalities  $[\zeta \cdot \eta_p]_O \geq [\xi \cdot \eta_p]_O$  for any  $p \leq q$  with  $\eta_p : f_p = 0$ . We will use known properties of the ultrametric  $d_{\mathcal{C}}$  distance (introduced in [20]) defined over the space  $\mathcal{C}$  of plane branches as  $\frac{1}{d_{\mathcal{C}}(C, D)} = \frac{[C \cdot D]_O}{e_O(C) e_O(D)}$  for any  $C, D \in \mathcal{C}$ . Hence, the inequalities above are equivalent to  $d_{\mathcal{C}}(\xi, \eta_p) \geq d_{\mathcal{C}}(\gamma, \eta_p)$  for any  $p \leq q$ , since in our case

$$\frac{[\zeta \cdot \eta_p]_O}{e_O(\zeta) e_O(\eta_p)} = \frac{n_{g-1} [\gamma \cdot \eta_p]_O}{n_{g-1} e_O(\gamma) e_O(\eta_p)} = \frac{1}{d_{\mathcal{C}}(\gamma, \eta_p)}.$$

Notice that  $f_q = f_{q_g}$  for some point  $q_g$ ,  $O \leq q_g \leq q$ , which corresponds to a dead-end in the dual graph of  $B_q$ . Now we summarize the results on the ultrametric space of plane branches of [1, 3.1, 3.2 and 3.4] adapted to our setting:

- $d_{\mathcal{C}}(\xi, \eta_p) = d_{\mathcal{C}}(\gamma, \eta_p)$ , if  $O \leq p < q_g$ ;
- $d_{\mathcal{C}}(\xi, \eta_p) = d_{\mathcal{C}}(\gamma, \eta_p)$ , if  $q_g < p \leq q$  and in the dual graph of  $B_q$  the vertex of  $p$  lies on the segment joining the vertexes  $q$  and  $q_g$ ;
- $d_{\mathcal{C}}(\xi, \eta_p) > d_{\mathcal{C}}(\gamma, \eta_p)$ , otherwise.

Hence, the second claim follows.  $\square$

**Proposition 3.4.** *Using the notations in Algorithm 2.3, at any step (i) of the algorithm, there exists a (power of) a maximal contact element  $f_{\tau_j}^{n_j} \in \mathcal{O}_{X,O}$  as required at step (i.j.3) and such element belongs to  $H_{B_{q_j}}$  but not to  $H_{\widehat{D}_j}$ .*

*Proof.* We are going to break the proof of the first statement in two cases depending on whether the point  $q_j \in K$  is free or satellite. With the notations from step (i.j.3),  $p \in K$  will be the last free point such that  $e_p(B_{q_j}) \neq 0$ .

Assume first that  $q_j$  is free, i.e.  $p = q_j$ . If, in addition, the vertex of  $q_j$  is a dead-end of the dual graph of  $F$  we are done, since  $f_{\tau_j} = f_{q_j}$  and  $f_{q_j} \in H_{B_{q_j}}$ . If  $q_j$  is not a dead-end of the dual graph, there is a dead-end  $q \in K$  and a totally ordered sequence  $q_j \leq p_1 \leq \dots \leq p_r \leq q$  of free points such that  $e_{q_j}(B_q) = e_{p_1}(B_q) = \dots = e_{p_r}(B_q) = e_q(B_q) = 1$ , by (1.4). Therefore,  $f_{\tau_j} = f_q$  with  $e_{q_j}(f_q) = 1$  and  $B_q = B_p + \overline{E}_{p_1} + \dots + \overline{E}_{p_r} + \overline{E}_q$ , which implies, by (1.1), that  $f_q \in H_{B_q} \subsetneq H_{B_{q_j}}$ . In either case we have that  $n_j = e_O(B_{q_j})/e_O(f_{\tau_j}) = 1$ .

Now, assume that  $q_j$  is satellite and hence  $p < q_j$ . Let  $\Sigma(\xi) = \langle n, \check{m}_1, \dots, \check{m}_g \rangle$  the semigroup of  $\xi : f_{q_j} = 0$ . By [8, §5.8],  $p$  has the property that any  $\gamma : f_p = 0, f_p \in \mathcal{O}_{X,O}$  satisfies  $[\gamma.\xi]_O = B_p \cdot B_{q_j} = \check{m}_g$ . However, it may happen that  $p \in K$  is not a dead-end. In this case, using the same argument as before, there is a dead-end  $q \in K$  and a totally ordered sequence  $p \leq p_1 \leq \dots \leq p_r \leq q$  of free points and  $B_q = B_p + \overline{E}_{p_1} + \dots + \overline{E}_{p_r} + \overline{E}_q$ . Since  $p$  is the last free point of  $B_{q_j}$ ,  $B_{q_j} \cdot \overline{E}_{p_i} = 0$  for  $i = 1, \dots, r$  and also  $B_{q_j} \cdot \overline{E}_q = 0$ . Hence,  $[\tilde{\gamma}.\xi]_O = B_q \cdot B_{q_j} = \check{m}_g$  with  $\tilde{\gamma} : f_q = 0$ , i.e. we can take  $f_{\tau_j} = f_q$ . We can then apply Lemma 3.3 to  $B_{q_j}$  with  $f_g = f_{\tau_j}$  yielding that  $f_{\tau_j}^{n_j} \in H_{B_{q_j}}$  with  $n_j = e_O(B_{q_j})/e_O(f_{\tau_j})$ .

Finally, if  $f_{\tau_j}^{n_j} \in \mathcal{O}_{X,O}$  fulfills the requirements of step (i.j.3), then  $e_O(f_{\tau_j}^{n_j}) = e_O(B_{q_j})$ , but  $e_O(\widehat{D}_j) > e_O(B_{q_j})$  by Proposition 2.1, therefore we have that  $f_{\tau_j}^{n_j} \notin H_{\widehat{D}_j}$ .  $\square$

**Theorem 3.5.** *Let  $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$  be an ideal. Then, Algorithm 2.3 computes a set of generators for the integral closure  $\overline{\mathfrak{a}}$  that are monomials in a given set of maximal contact elements of  $\mathfrak{a}$ .*

*Proof.* Let us prove that the  $i$ -th step of the algorithm returns a system of generators of  $H_{D^{(i)}}$  which has the desired properties. By Zariski's factorization Theorem 1.5, it is enough to focus on computing generators for each simple ideal  $H_{B_{q_j}}, j = 1, \dots, d_i$ , in the decomposition of  $D^{(i)}$ . Fixing  $B_{q_j}$  at step (i.2), we will make induction on the order of the neighbourhood that  $q_j \in K$  belongs to, and we will show that Algorithm 2.3 computes generators for  $H_{B_{q_j}}$  which are monomials in the set of maximal contact elements.

If  $q_j = O$ , then  $B_O = \text{Div}(\pi^*\mathfrak{m})$  and step (i.j.1) returns  $H_{D^{(i)}} = \mathfrak{m}$ , since a pair of smooth transverse elements generate  $\mathfrak{m}$ . By construction, any set of maximal contact elements contain such a pair of elements.

Assume now that  $q_j \neq O$  and that the algorithm computes the generators of the ideals associated to  $B_p$  for  $p < q_j$ . By Proposition 2.1,  $H_{\widehat{D}_j} \subsetneq H_{B_{q_j}}$  are adjacent ideals. Since  $\widehat{D}_j = \sum_{p < q_j} \rho_p^{(i)} B_p$ , we can apply the induction hypothesis to the simple divisors  $B_p, p < q_j$  such that  $\rho_p^{(i)} \neq 0$  and apply Theorem 1.5 to get

$$H_{\widehat{D}_j} = \prod_{p < q_j} H_{B_p}^{\rho_p^{(i)}} \subsetneq H_{D^{(i)}}.$$

At this point it is enough to add any element that belongs to  $H_{B_{q_j}}$  but not to  $H_{\widehat{D}_j}$  to get a system of generators of  $H_{B_{q_j}}$ . By Proposition 3.4, the element chosen at step (i.j.3) has the desired properties, namely, it is a power of a maximal contact element. Finally, we can remove unnecessary elements from the system of generators of  $H_{D^{(i)}}$  using Lemma 2.2.  $\square$

**Corollary 3.6.** *The monomial expression in the maximal contact elements of the generators of  $\bar{\mathfrak{a}}$  given by Algorithm 2.3 is independent of the set of maximal contact elements chosen.*

*Proof.* The conditions required on an maximal contact elements  $f_p$ ,  $p \in K$  in step (i.j.3) of Algorithm 2.3 depend only on the multiplicities of this element in points  $q \leq p$ ,  $q \in K$  and hence only in its log-resolution.  $\square$

**Corollary 3.7.** *Let  $\pi : X' \rightarrow X$  be a sequence of blow-ups. Any complete ideal  $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$  whose log-resolution is dominated by  $\pi$  admits a system of generators given by monomials in a fixed set of maximal contact elements associated to  $\pi$ .*

*Remark 3.8.* From Theorem 5 of [11], applied to the set of divisorial valuations associated to all the points blown-up in  $\pi$ , one may deduce the existence of a set of maximal contact elements such that any complete ideal whose log-resolution is dominated by  $\pi$  is generated by monomials in those maximal contact elements. Corollary 3.7 recovers this result in this particular case and provides an explicit construction of these generators.

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